Phase Transitions and Critical Behavior
for Charged Black Holes

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Abstract

We investigate the thermodynamics of a four-dimensional charged black hole in a finite cavity in asymptotically flat and asymptotically de Sitter space. In each case, we find a Hawking-Page-like phase transition between a black hole and a thermal gas very much like the known transition in asymptotically anti-de Sitter space. For a "supercooled" black hole—a thermodynamically unstable black hole below the critical temperature for the Hawking-Page phase transition—the phase diagram has a line of first-order phase transitions that terminates in a second order point. For the asymptotically flat case, we calculate the critical exponents at the second order phase transition and find that they exactly match the known results for a charged black hole in anti-de Sitter space. We find strong evidence for similar phase transitions for the de Sitter black hole as well. Thus many of the thermodynamic features of charged anti-de Sitter black holes do not really depend on asymptotically anti-de Sitter boundary conditions; the thermodynamics of charged black holes is surprisingly universal.

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1 Introduction

It has been known for some time that a black hole in asymptotically anti-de Sitter space can undergo a phase transition, "evaporating" at a critical temperature into a hot gas [1]. This Hawking-Page phase transition has recently garnered new attention with the realization that it can be used to test the AdS/CFT correspondence by identifying a similar structure in a dual conformal field theory [2]. A dual description may exist not only for this particular transition, but for a number of finer details of the phase structure of the asymptotically AdS black hole as well [3,4]. The purpose of this paper is to show that this phase structure is not unique to the asymptotically AdS black hole, but occurs universally for suitably stabilized black holes in asymptotically flat and asymptotically de Sitter space.

A black hole in anti-de Sitter space can be thermodynamically stable. In asymptotically flat or de Sitter space, this is no longer the case: an isolated black hole radiates away its energy in the form of Hawking radiation. To understand equilibrium black hole thermodynamics, one must therefore work with ensembles that include not just the black hole, but also its environment [5]. As self-gravitating systems are spatially inhomogeneous, any specification of such ensembles requires not just thermodynamic quantities of interest, but also the place at which they take the specified values. Having set up an appropriate thermodynamic ensemble, one can then proceed to ask interesting questions regarding stability and phase structure.

Such an analysis was done for the case of charged black holes in asymptotically flat space by Braden, Brown, Whiting, and York [6]. Following a brief summary of their analysis, we will build on those results to study the phase structure of charged black holes. We will also apply a similar method to charged black holes in de Sitter space. This work grew out of an effort to understand the effect of varying "constants" on black hole thermodynamics [7], but it may have broader significance. In particular, the universality of the phase structure may have interesting implications for holography in asymptotically flat and de Sitter space.

The Euclidean action [8] for metric g and electromagnetic field A_μ over a region M with boundary ∂M takes the form

$$I = -\frac{1}{16\pi} \int_M d^4x \sqrt{g}\left[(R - 2\Lambda) - F^2\right] + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{\gamma}\,(K - K_0), \qquad (1.1)$$

where the cosmological constant Λ is zero for flat space and positive for de Sitter space. The geometric and electromagnetic data specified at the boundary ∂M fixes for us the thermodynamic data for this gravitational system thought of as a statistical ensemble. We take a spherically symmetric ansatz for the metric of a black hole spacetime,

$$ds^2 = f^2 d\tau^2 + \alpha^2 dy^2 + r^2 d\Omega^2, \qquad (1.2)$$

where f, α and r are functions of the radial coordinate $y \in [0, 1]$. The reduced action may be calculated from (1.1) provided certain conditions are taken into account:

1. The horizon of the black hole is located at $y = 0$ (i.e. $r(0) = r_+$), and we restrict to the class of metrics that are "regular." In other words, the geometry of the spacetime near the origin of the y-τ plane looks like the plane $\mathbb{R}^2$, so the horizon is nondegenerate.

2. The boundary at $y = 1$ has a topology $S^1 \times S^2$ with the two-sphere having an area $4\pi r_B^2$. Heat can flow in either direction through this boundary in such a manner as to keep the temperature β^{-1} fixed. The inverse temperature is simply the proper length of the circle S^1 of the boundary: $\beta = 2\pi f(1)$ (where we take τ to have period 2π).

We also need to specify appropriate electromagnetic data at the boundary $y = 1$. For the grand canonical ensemble, we specify the gauge potential $A_\tau(1)$ (more precisely the difference of potential between $y = 0$ and $y = 1$). For the canonical ensemble, we must add a boundary term to the usual Maxwell action and hold the electromagnetic field fixed at the boundary. This is equivalent to keeping the total charge inside the cavity fixed.

Using this analysis, we first demonstrate in section 2 that black holes in asymptotically flat space show evidence of a Hawking-Page phase transition to a thermal gas. While the transition temperature depends on the charge, the variation is surprisingly small. In the absence of a well-understood comparison action for a charged gas, conclusions from (fixed-charge) canonical ensemble are somewhat ambiguous; we therefore also check for this transition in the grand canonical ensemble, and confirm its existence.

We next investigate the unstable "supercooled" region of the phase diagram, and show that black holes in this region, like those in anti-de Sitter space [3, 4], exhibit a first order phase transition: for a given (small) value of charge, the entropy of the system changes discontinuously as the temperature is varied. As the charge increases, the "height" of this discontinuity decreases, and the line of first order phase transitions terminates at a critical point. We show that this critical point is the location of a second order phase transition, and we calculate the critical exponents associated with it. Remarkably, the critical exponents are the same as those found by Chamblin et al. [3,4] for charged black holes in anti-de Sitter space, hinting that these two systems belong to the same universality class.

In section 3, we apply the reduced action technique to charged black holes in de Sitter space. We again find evidence for a Hawking-Page phase transition. For sufficiently small cosmological constant, the supercooled region again displays a line of first order phase transitions terminating at a critical point. This picture no longer remains valid as Λ increases. In particular, it seems that for large values of Λ, there is no second order transition: the location of the would-be transition is pushed beyond the allowed boundaries of the cavity.

The reduced action for the de Sitter case is more complicated, and we cannot find the critical exponents of the second order phase transition (when it exists) analytically. We can study two special cases: the uncharged black hole for any value of Λ and

the charged black hole for small Λ. We again find the same structure, with critical exponents that are unchanged by the presence of a small cosmological constant.

Section 4 is devoted to discussion our results.

2 Charged black hole in asymptotically flat space

Following Brown et al. [6], we first insert the ansatz (1.2) into the action (1.1), and find that the Hamiltonian constraint can be solved, yielding

$$V(r) \equiv \left(\frac{r'}{\alpha}\right)^2 = 1 - \frac{C}{r} + \frac{e^2}{r^2}. \tag{2.1}$$

For a nondegenerate horizon, $C = r_+ + e^2/r_+$, with $e^2 < r_+^2$. The reduced action for the canonical ensemble then takes the form

$$I_C = \beta r_B \left(1 - \sqrt{\left(1 - \frac{r_+}{r_B}\right)\left(1 - \frac{e^2}{r_+ r_B}\right)}\right) - \pi r_+^2. \tag{2.2}$$

Extremizing the action with respect to r_+ gives us

$$\beta = 4\pi r_+ \left(1 - \frac{e^2}{r_+^2}\right)^{-1} \left(1 - \frac{r_+}{r_B}\right)^{1/2} \left(1 - \frac{e^2}{r_+ r_B}\right)^{1/2}, \tag{2.3}$$

which should be viewed as an equation for the unknown r_+ in terms of the fixed charge, boundary radius, and temperature of the canonical ensemble. Each extremum gives a saddle point contribution to the Euclidean path integral for the canonical partition function $Z[\beta]$, with a Helmholtz free energy and entropy

$$
\begin{aligned}
F &= -\frac{1}{\beta}\ln Z[\beta] = \frac{1}{\beta}I_C[\beta, r_+(\beta)] \\
S &= -\left(\beta\frac{\partial}{\partial\beta} - 1\right)\ln Z[\beta] = \pi r_+^2.
\end{aligned}
\tag{2.4}
$$

Equation (2.3) can be rewritten as the seventh order algebraic equation

$$x^5(x-1)(x-q^2) + b^2(x^2-q^2)^2 = 0 \tag{2.5}$$

with $x = r_+/r_B, q = e/r_B$ and $b = \beta/4\pi r_B$. We are interested only on those solutions that are real positive and satisfy $q^2 < x^2$. Of these positive solutions, we initially choose the one that minimizes the free energy $F[\beta]$, since this is the thermodynamically stable state in the canonical ensemble.
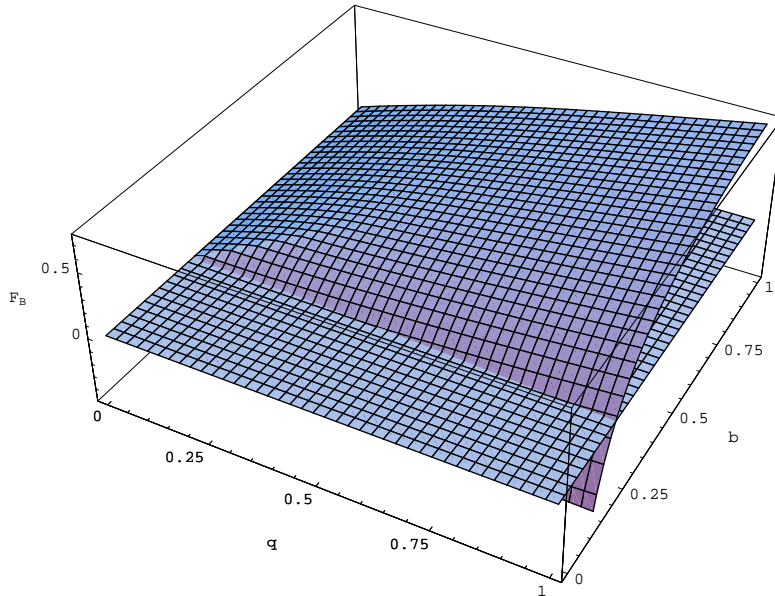
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Figure 1: Free energy of a charged black hole in asymptotically flat space as a function of inverse temperature b and charge q.

2.1 The Hawking-Page transition

The plot of free energy as a function of charge q and inverse temperature β is shown in Figure 1. For reference, we have also plotted the plane $F = 0$. In the original paper of Hawking and Page [1], the phase transition to a hot gas of particles was determined by comparing the black hole free energy with that of a real thermal gas, but subsequent work (for example, [2]) has often used "hot empty AdS" as a reference. This alternative is not available for a charged black hole in the canonical ensemble, except at $q = 0$, since there is no "hot empty charged space"; one cannot avoid the intricacies of a real charged gas.

Although there has been a bit of work on the thermodynamics of self-gravitating gases of particles [9, 10, 11], the subject remains rather poorly understood. For low temperature, Refs. [9, 10, 11] confirm the natural guess that the free energy is approximately Nm, where N is the number of gas particles of mass m and charge q_0, while the charge is $e = Nq_0$.* For real elementary particles, $q_0 \gg m$ in geometrized units— for an electron, for instance, $q_0/m \sim 10^{21}$—so $F \ll e$, even for a near-extremal black hole. An approximation $F = 0$ for a thermal gas thus seems reasonable, and the intersection of the $F = 0$ plane with the free energy in Figure 1 should give the rough location of a Hawking-Page-type phase transition. It is straightforward to check that above the transition temperature, the black hole is thermodynamically stable, i.e., $C_V = -b(\partial S/\partial b)$ is positive everywhere. Figure 2, for example, plots the entropy
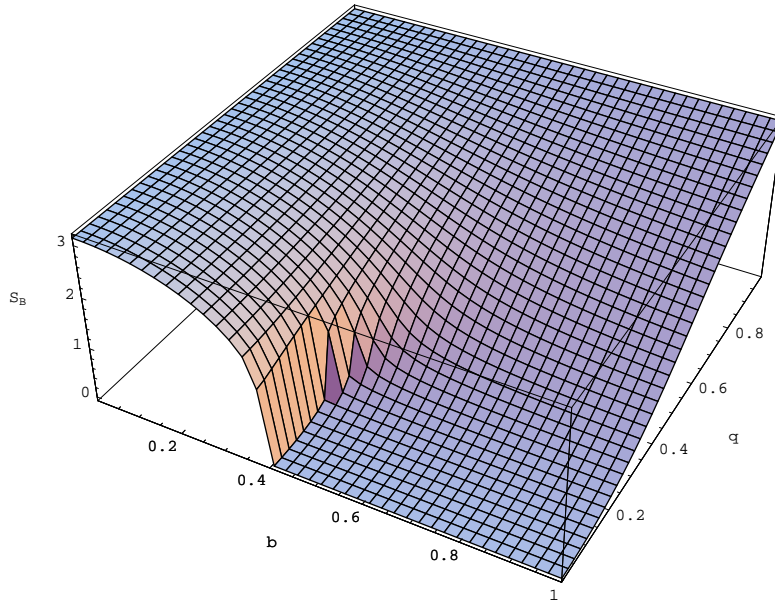
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*For Newtonian results, see also [12].

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Figure 2: Entropy of a charged black hole in asymptotically flat space as a function of inverse temperature b and charge q.

against q and β; the sign of C_V is determined by the slope.

In the absence of a better understanding of the thermodynamics of hot self-gravitating gases, this argument is admittedly not completely convincing. The "competing" configurations are, after all, not just dilute gases, but stars. To check more carefully for a Hawking-Page transition, we have therefore followed the suggestion of Ref. [13] and looked at the grand canonical ensemble, in which the boundary potential ϕ is fixed instead of the charge. This ensemble has the nice feature that it *does* allow a comparison with "hot empty space": while an empty cavity cannot contain charge, its walls can be held at a constant potential.

Figure 3 shows the Gibbs free energy of the grand canonical ensemble, with the plane $G = 0$ superimposed. The intersection shows the location of the Hawking-Page phase transition to hot empty space.[†] The transition depends only weakly on the potential. There is one important difference between the charged and uncharged cases, however. For an uncharged black hole, there is a further critical temperature beyond which the action has no extremum, even with positive free energy. Thus even a "supercooled" black hole cannot exist below this temperature. For a charged black hole, no such temperature exists; a locally stable black hole can exist for any value of β, essentially because a near-extremal black hole can have an arbitrarily low temperature.

[†]The Gibbs free energy becomes positive again at very high temperatures. This occurs when the thermodynamically stable black hole becomes larger than the cavity radius r_B, leaving only an unstable (negative specific heat) configuration.
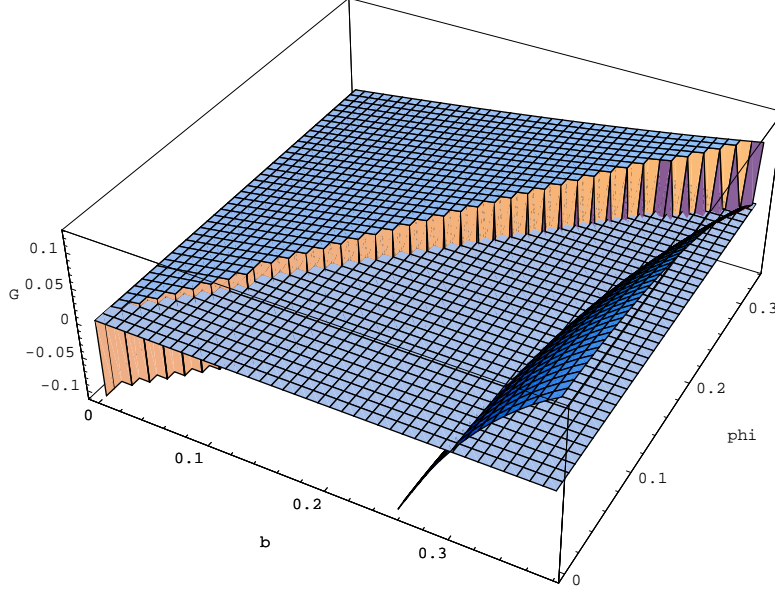
Figure 3: Gibbs free energy of a charged black hole in asymptotically flat space as a function of inverse temperature b and potential ϕ.

2.2 Supercooled black holes and critical exponents

Let us now return to the canonical ensemble, and consider the portion of Figure 1 for which the black hole free energy is positive. While a black hole in this region is globally unstable, it is locally stable—the specific heat is positive, as is evident from Figure 2. One can therefore produce a "supercooled" black hole in this region, and, indeed, the analysis of fluctuations for uncharged black holes suggests that such a state may have a very long lifetime [14, 15]. Figure 1 now suggests that there is a line of first order phase transitions in this region that terminates at a critical point in the (q, β)-plane. We will show that this critical point is the location of a second order phase transition.

The simplest way to see this is to rewrite (2.3) as

$$b(x, q) = \frac{x(1-x)^{1/2}(1-q^2/x)^{1/2}}{1-q^2/x^2}. \tag{2.6}$$

In general, $b(x, q)$ has may have extrema (both maxima and minima) as a function of x, with locations

$$\frac{\partial b}{\partial x} = \frac{x(q^4(5-6x) + (2-3x)x^3 + 2q^2x(-3+3x+x^2))}{2(q^2-x^2)^2\sqrt{1-x}\sqrt{1-q^2/x}} = 0. \tag{2.7}$$

The numerator is a polynomial in x (with functions of q as coefficients), with a discriminant

$$\Delta(q) = \frac{6400}{243}q^4(1-q^2)^4(1-18q^2+q^4). \tag{2.8}$$

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This discriminant determines the condition for the maximum and the minimum to coincide: they do so for $\Delta(q) = 0$, i.e., for $q_c = -2 + \sqrt{5}$. The corresponding critical values for the horizon radius and temperature are $x_c = 5 - 2\sqrt{5}$ and $b_c = \frac{5}{2}(17 + 38/\sqrt{5})^{-1/2}$. Thus $(x_c, b_c, q_c) \simeq (0.528, 0.429, 0.236)$. Expanding in the neighborhood of the critical point, we obtain

$$b - b_c = -\frac{1}{8}(425 + 195\sqrt{5})^{1/2}(x - x_c)^3 + \cdots \simeq -3.644(x - x_c)^3 + \cdots . \qquad (2.9)$$

To see the dependence on x near q_c, we rewrite (2.3) as

$$q(x, b) = \frac{(2b^2 x^2 - x^5 + x^6 - (1-x)x^3\sqrt{4b^2 + x^4})^{1/2}}{\sqrt{2}b} \qquad (2.10)$$

and expand in the neighborhood of (x_c, b_c):

$$q - q_c = \frac{1}{40}(105 + 47\sqrt{5})(x - x_c)^3 + \cdots \simeq 5.252(x - x_c)^3 + \cdots . \qquad (2.11)$$

Inverting these relations and writing in terms of the entropy S, we find

$$S - S_c = 1.226(\tau - \tau_c)^{1/3} + \cdots , \qquad (2.12)$$
$$S - S_c = 41.908(q - q_c)^{1/3} + \cdots , \qquad (2.13)$$

where $\tau = r_B kT, \tau_c = 1/b_c$ and $S_c = \pi x_c^2$. The specific heat $C_v = T(\partial S/\partial T) = \tau(\partial S/\partial \tau)$ is

$$C_v \simeq 0.409(\tau - \tau_c)^{-2/3} \qquad (2.14)$$

which means that the critical exponent α is $-2/3$. Remarkably, this is also the critical behavior of the charged black hole in asymptotically anti-de Sitter space [3, 4, 18]. There, as in the asymptotically flat case discussed here, a line of first order phase transitions in the "supercooled" region ends in a second order transition, and although the AdS partition function has a completely different functional form, the critical exponents are identical.

The case of uncharged black hole is even simpler. Equation (2.5) reduces to the cubic $x^3 - x^2 + b^2 = 0$, with critical values $(x_c, b_c) = (2/3, 2/(3\sqrt{3}))$. This critical point does not signal a second order phase transition [6]: in contrast to the case of even arbitrarily small charge, the action for the uncharged black hole has no extremum beyond this point. As discussed at the end of section 2.1, this point instead describes a transition between even a supercooled black hole and a thermal gas. Near this point,

$$S - S_c \sim (\tau - \tau_c)^{1/2} \qquad (2.15)$$

indicating that the specific heat diverges like $C_v \sim (\tau - \tau_c)^{-1/2}$. This is that same critical behavior seen in non-spinning $D3$-branes [17, 19].

3 Charged black holes in de Sitter space

Conceptually, the method of analyzing the thermodynamic stability of black holes in de Sitter space is the same as in flat space. Lorentzian de Sitter space has infinite spatial extent, but in static coordinates there is a cosmological horizon. The corresponding Euclidean section of the metric has finite volume, with topology of a four-sphere with a radius equal to the cosmological horizon.

For a charged black hole in de Sitter space, the spacetime has, generically, three horizons: an inner and an outer horizon for the black hole, and the cosmological horizon. Any discussion of equilibrium black hole thermodynamics requires one to specify thermodynamic quantities at a boundary that is inside the cosmological horizon. In this case, the entropy of the equilibrium configuration depends on three parameters: temperature, charge, and the cosmological constant. Again using the spherically ansatz (1.2) in the action and solving the Hamiltonian constraint, we find

$$V(r) = \left(\frac{r'}{\alpha}\right)^2 = 1 + \frac{e^2}{r^2} - \frac{\Lambda r^2}{3} - \frac{C}{r}. \tag{3.1}$$

The integration constant C is often identified as the mass parameter $2M$.

Our first task is to determine the allowed values of the parameters. For the path integral to be well-defined, we need to restrict to those values of e, M, Λ and r for which $V(r)$ is nonnegative. Event horizons are located at those values of r for which $V(r) = 0$. To locate these, it is useful to define dimensionless quantities $\rho \equiv r\sqrt{\Lambda}, g \equiv e\sqrt{\Lambda}$ and $\mu \equiv M\sqrt{\Lambda}$, so

$$V = -\frac{1}{3\rho^2}Q(\rho; \mu, g^2) = -\frac{1}{3\rho^2}(\rho^4 - 3\rho^2 + 6\mu\rho - 3g^2) \tag{3.2}$$

The quartic polynomial Q has four real roots: one negative root, and three positive roots that we denote, in ascending order, $\rho_-(\mu, g^2)$, $\rho_+(\mu, g^2)$, and $\rho_c(\mu, g^2)$. The region outside the outer horizon of the black hole is thus $\rho_+(\mu, g^2)/\sqrt{\Lambda} \le r \le \rho_c(\mu, g^2)/\sqrt{\Lambda}$. The condition for two (or more) of these roots to coincide [16] is given by the vanishing of the discriminant of $Q(\rho, \mu, g^2)$:

$$\Delta(\mu, g^2) = (1 - 4g^2)^3 - (1 + 12g^2 - 18\mu^2)^2 = 0. \tag{3.3}$$

The the allowed values of mass and charge parameter in the (μ, g)-plane are thus given by the interior of the region bounded by the curves

$$\mu_{min}(g) = \sqrt{\frac{1 + 12g^2 - \sqrt{(1 - 4g^2)^3}}{18}}, \tag{3.4}$$

$$\mu_{max}(g) = \sqrt{\frac{1 + 12g^2 + \sqrt{(1 - 4g^2)^3}}{18}}, \tag{3.5}$$

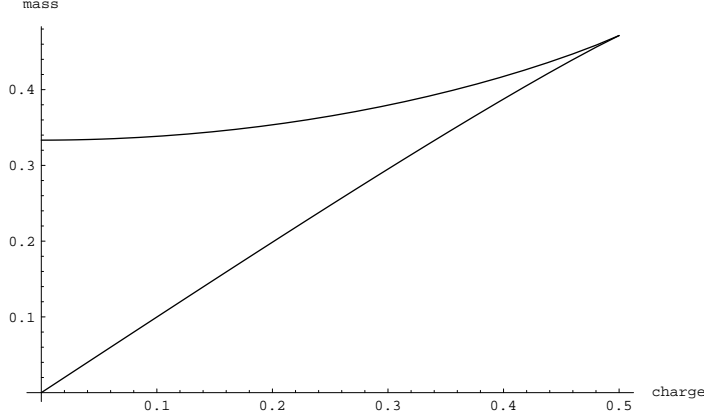which intersect at the point $(\mu, g) = (\sqrt{2}/3, 1/2)$. The curve $\mu_{min}(g)$ corresponds

Figure 4: Allowed values of mass μ and charge g.

to $r_- = r_+$, i.e., coincident inner and outer horizons, while the curve $\mu_{max}(g)$ corresponds to $r_+ = r_c$. These two conditions can be combined into a simple form $g^2 = \rho^2(1 - \rho^2)$.

The de Sitter black hole action is thus a function of four variables, μ, g, $b \equiv \beta\sqrt{\Lambda}$, and the box size $R \equiv r_B\sqrt{\Lambda}$. To locate a saddle point, we must find the roots of $V(r) = 0$, choose the middle positive root, substitute it into the action, and minimize it, subject to the condition that μ and g lie in the interior of the region depicted in Fig. 4.

To analyze the stability of the black hole, it is convenient to revert to the variables r_+ and e rather than their "normalized" versions μ and g. We will also stay away the extremal black holes corresponding to the curves $\mu_{min}(g)$ and $\mu_{max}(g)$, by requiring that $g^2 < \rho_+^2(1 - \rho_+^2)$ and $g^2 < \rho_c^2(1 - \rho_c^2)$. Requiring that the r-τ plane look like $\mathbb{R}^2$ near $r = r_+$ fixes C to be $r_+ + e^2/r_+ - \Lambda r_+^3/3$, which puts $V(r)$ in the form

$$V(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{e^2/r_+}{r} - \frac{\Lambda}{3}(r^2 + rr_+ + r_+^2)\right). \tag{3.6}$$

Thermodynamic quantities are specified at a boundary $r = r_B$, and we require that $r_B < r_c$. For any given value of r_+ and e, the maximum allowed value of r_B is given by the larger root of the equation $g^2 = \rho^2(1 - \rho^2)$. To summarize, for any given value of $x = r_+/r_B$, the allowed values of charge $q = e/r_B$ are given by $q^2 < x^2(1 - R^2x^2)$ and $q^2 < 1 - R^2$.

Subject to the above conditions, the reduced action can be shown to be

$$
\begin{aligned}
I_C &= \beta r_B[1 - \sqrt{V(r_B)}] - \pi r_+^2 \tag{3.7}\\
&= \beta r_B\left[1 - \sqrt{\left(1 - \frac{r_+}{r_B}\right)\left(1 - \frac{e^2}{r_Br_+} - \frac{\Lambda}{3}(r_B^2 + r_Br_+ + r_+^2)\right)}\right] - \pi r_+^2. \tag{3.8}
\end{aligned}
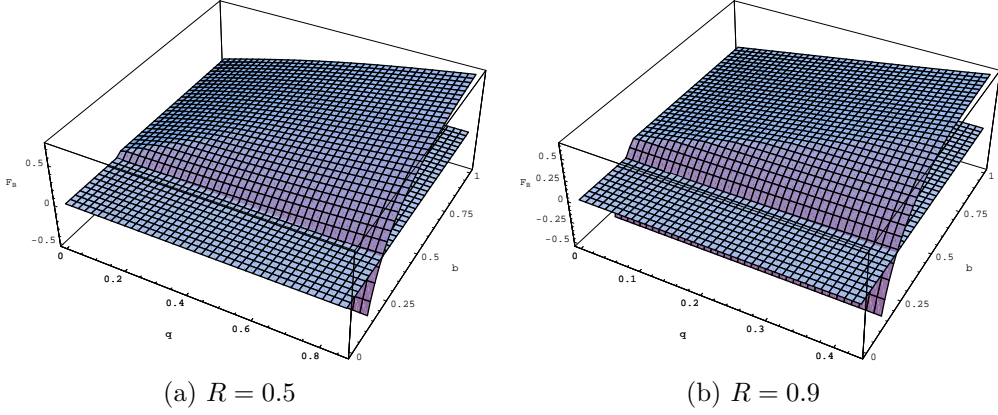$$

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(a) $R = 0.5$ (b) $R = 0.9$

Figure 5: Helmholtz free energy of the black hole for two values of R.

Extremizing that action with respect to x and solving for b, we obtain

$$\beta = \frac{4\pi r_+ \sqrt{\left(1 - \frac{r_+}{r_B}\right)\left(1 - \frac{e^2}{r_+ r_B} - \frac{\Lambda}{3}(r_B^2 + r_+ r_B + r_+^2)\right)}}{1 - \frac{e^2}{r_+ r_B} - \frac{\Lambda}{3}(r_B^2 + r_+ r_B + r_+^2) + \left(1 - \frac{r_+}{r_B}\right)\left(\frac{\Lambda}{3}(r_B^2 + 2r_+ r_B) - \frac{e^2}{r_+^2}\right)}. \tag{3.9}$$

This is a ninth order algebraic equation for r_+, which may be written as

$$x^5(1 - x)\left(x - q^2 - \frac{R^2 x(1 + x + x^2)}{3}\right) - b^2(q^2 - x^2 + R^2 x^4)^2 = 0. \tag{3.10}$$

Given values for b, q and R, (3.10) can be solved numerically for x. In Figure 5, we plot the free energy as a function of charge q and inverse temperature b for two values of the cavity radius R. As before, we also show the $F = 0$ plane, which should give an approximate locus for the Hawking-Page phase transition.

As in the asymptotically flat case, we have checked this behavior in the grand canonical ensemble. Figure 6 shows the Gibbs free energy and the $G = 0$ plane, and confirms the existence of a phase transition. Note that a Hawking-Page phase transition occurs for all values of Λ. While the presence of a cosmological constant affects the transition temperature, it does not change the qualitative behavior.

For small values of R, it is clear that the charged black hole in de Sitter space behaves very much like the corresponding black hole in flat space. There is again a line of first order phase transitions in the supercooled region that ends at a critical point. For large values of R, the critical point disappears: the place at which it would occur is essentially pushed outside the cosmological horizon.

In general, we have been unable to find the nature of the phase transition at the critical point. We can, however, do so for for two important special cases: the uncharged black hole ($e = 0$) and the case of small cosmological constant. For $e = 0$,
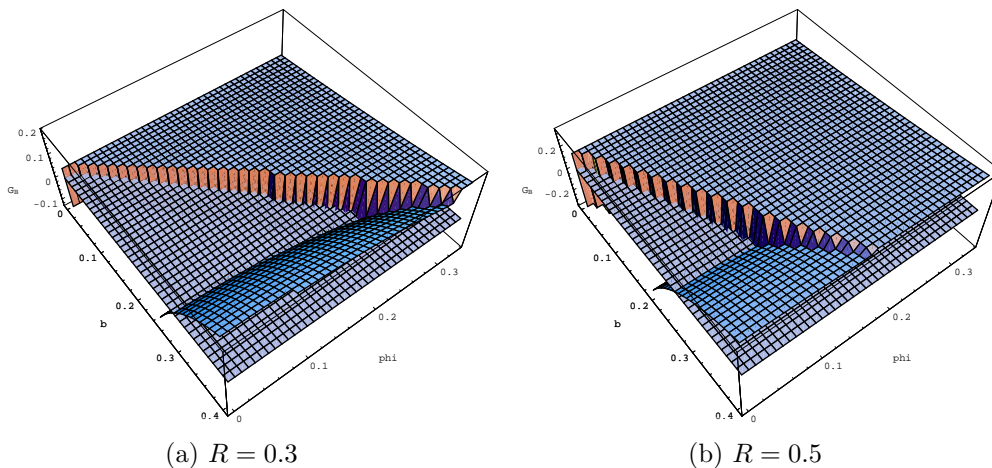
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(a) $R = 0.3$ (b) $R = 0.5$

Figure 6: Gibbs free energy of the black hole for two values of R.

the function b takes the form

$$b(x, R) = \frac{x\sqrt{(1-x)(1 - R^2(1 + x + x^2)/3)}}{1 - R^2 x^2}. \tag{3.11}$$

The positive root of the equation $1 - \Lambda(r^2 + r_+ r + r_+^2)/3 = 0$ gives us the location of the cosmological horizon r_c. Requiring that $r_+ < r_c$ tells us that $r_+^2 < 1/\Lambda$. Similarly, $r_+ < r_B < r_c$ implies that $R \equiv r_B \sqrt{\Lambda} < 1$.

The free energy can be plotted as a function of b and R (Figure 7), and clearly shows that black holes do not exist above a critical value b_c of b, for any fixed value of R. This critical value is analogous to the value discussed at the end of section 2.1: it is a limiting temperature beyond which not even a supercooled black hole can exist. For any given value of R, the entropy of the system behaves as $S - S_c(R) \sim (b - b_c(R))^\alpha$. By choosing various values of R, one can check that $\alpha = 1/2$ and that $C_v \sim (\tau - \tau_c)^{-1/2}$. The critical behavior thus is very much like that of the uncharged black hole in flat space discussed in section 2.2.

For the case of small Λ, we can expand b in powers of R^2 and look for critical points in the supercooled regime. Both b_c and q_c receive correction to order R^2, while x_c remains the same to this order. The critical exponents also remain the same.

4 Discussion

The thermodynamics of black holes is interesting in its own right, and also because of its connections to holography and the AdS/CFT correspondence. Largely inspired by the latter connection, past investigations have found a rich phase structure for asymptotically anti-de Sitter black holes.

Our main result is that this phase structure does not, in fact, require asymptotically anti-de Sitter boundary conditions, but appears as well for black holes in
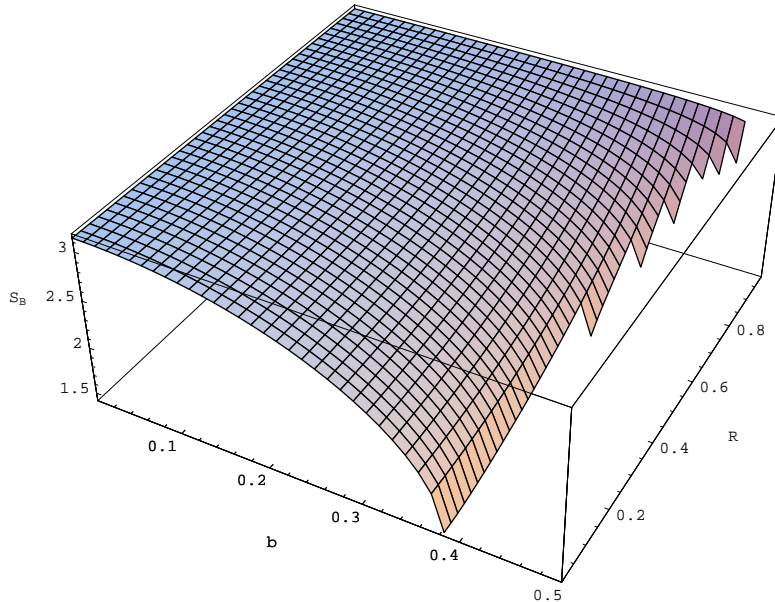
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Figure 7: Free energy for the uncharged de Sitter black hole as a function of inverse temperature b and cosmological constant $R = \sqrt{\Lambda} r_B$.

cavities in asymptotically flat and asymptotically de Sitter spaces. In particular, both Hawking-Page-like phase transitions and transitions between small and large black holes occur as we vary the charge and temperature. For "supercooled" black holes in flat space, and black holes in de Sitter space with small Λ, we found a line of first order phase transitions ending at a second order transition, and we were able to identify the critical exponents at the second order point. Remarkably, these were the same as those for asymptotically anti-de Sitter black holes, despite a very different algebraic structure of the partition function.

For anti-de Sitter space, the phase structure of charged black holes has served as a valuable test of the AdS/CFT correspondence. We leave open the very interesting question of whether the corresponding results for asymptotically flat and asymptotically de Sitter black holes might cast light on proposals for finite-volume holography in this more general setting.

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